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ETS IV: Sequential Dynamical Systems: Fixed Points, Invertibility and Equivalence

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Abstract

Sequential dynamical systems (SDS) are discrete dynamical systems that are obtained from the following data: (a) a finite (labeled) graph Y with vertex set $\{1, \dots, n\}$ where each vertex has a binary state, (b) a vertex labeled sequence of functions $(F_{i,Y} : \mathbb{F}_2^n \to \mathbb{F}_2^n)_i$ and (c) a permutation $\pi \in S_n$. The function $F_{i,Y}$ updates the binary state of vertex i as a function of the states of vertex i and its Y-neighbors and leaves the states of all other vertices fixed. The permutation π represents a Y-vertex ordering according to which the functions $F_{i,Y}$ are applied. By composing the functions $F_{i,Y}$ in the order given by π we obtain the sequential dynamical system (SDS)

$$[F_Y, \pi] = \prod_{i=1}^{n} F_{\pi(i),Y} : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n \ .$$

In this paper we will generalize a class of results on SDS that have been proven for symmetric Boolean (local) functions to quasi-symmetric local functions. Further, we completely classify invertible SDS and investigate fixed points of sequential and parallel cellular automata (CA). Finally, we show sharpness of a combinatorial upper bound for the number of non-equivalent SDS that can be obtained through rescheduling for a certain class of graphs.

Key words: sequential dynamical system, quasi-symmetric functions, fixed points, digraph isomorphism, topological conjugation.

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1 Introduction

Sequential dynamical systems (SDS) [5, 7] are a class of discrete dynamical systems that were originally introduced in the context of formalizing computer simulations. This paper continues the line of research on theoretical foundations for simulations initiated in [3–5].

SDS directly captures the key constituents of a computer simulation: they are given by (a) a collection of local functions (agents or object libraries), (b) a graph (the interactions among agents) and (c) an update schedule (the order in which the agents act). In the SDS framework a computer simulation is viewed as a dynamical system, and generic questions such as classification, categorization and validations can be translated into mathematically precise questions on dynamical systems. In particular, the classification problem for simulations now becomes a question on non-isomorphism of phase spaces for the corresponding SDS. An immediate consequence of this identification is that many observables that are commonly used for the classification of simulations are often unsuited or insufficient for their purpose.

In a computer simulation one typically has "perfect" knowledge about each agent or entity and its communication capabilities in isolation. However, to retrieve information on the composed global dynamics produced through local interaction among agents, one will typically have to implement and run the whole simulation system. In fact, in view of [1] running a simulation is often the best thing one can do as it is impossible to find a computationally more effective description of the system in question.

The character of the results in this paper is to extract dynamical properties of SDS from known quantities such as the dependency graph and the update rules, without actually implementing and running the SDS on a computer. This is also the approach in the earlier work on discrete sequential dynamical systems [3, 5, 7, 9]. In this sense it is possible to obtain important information about a simulation and its global behavior based on "local" knowledge, that is, without actually performing computer runs.

A number of results have also been obtained on the computational complexity for the analysis of the behavior of SDS and the structure of their phase spaces [1, 2]. These results are centered around characterizations of the complexities

of state reachability problems for many classes of SDS. These reachability problems are shown to be PSPACE-hard by very efficient reductions of the membership problems for arbitrary deterministic linear space-bounded turning (Turing???) machines. The proofs of these PSPACE-hardness results also show the following:
(a) the class of SDS, even when restricted to so-called vertex functions that are identical and symmetric, and where the underlying graph is linear (a line???), can efficiently "simulate" the behavior of arbitrary finite systems/networks of CA, non-linear difference equations on finite algebraic structures and interconnected finite automata.
(b) the "simulating" SDS can be constructed very efficiently from the system/network to be "simulated" by local replacement.
The implications of these highly efficient simulations and translations, e.g. the efficient "universality" of SDS (for the finite discrete case) are under investigation.

In this paper we will first generalize the class of permissible local functions for SDS from symmetric to quasi-symmetric local functions. In this context we will revisit several key results and establish their validity for this generalized function class. Next we will investigate fixed points of sequentially and parallelly updated cellular automata (CA). This has been solved in [4] for elementary cellular automata and here we extend this framework to cellular automata with rules of arbitrary cell size. Finally, we prove the sharpness of a combinatorial bound for the number of non-equivalent sequential dynamical systems that can obtained through rescheduling [7, 9] for a certain graph class using Boolean nor functions as local maps.

To make the paper self-contained let us revisit the basic framework of SDS which we immediately will generalize to quasi-symmetric functions:
Let Y be a labeled graph with vertex-set $\mathrm{v}[Y] = \mathbb{N}_n = \{1, 2, 3, \ldots, n\}$. We write this as $Y < K_n$. The edge-set of Y is denoted by $\mathrm{e}[Y]$. Let $S_{1,Y}(i)$ be the set of Y-vertices that are adjacent to vertex i and let $\delta_i = |S_{1,Y}(i)|$. The increasing sequence of elements of the set $\{i\} \cup S_{1,Y}(i)$ is denoted by

$$\tilde{B}_{1,Y}(i) = (j_1, \ldots, i, \ldots, j_{\delta_i}). \tag{1}$$

The maximal degree in Y is $d = \max_{1 \leq i \leq n} \delta_i$.

To each vertex i we associate a state $x_i \in \mathbb{F}_2$, and for each $k = 1, \ldots, d+1$

we have a symmetric function $f_k : \mathbb{F}_2^k \to \mathbb{F}_2$. For each vertex i we introduce a map

$$\mathrm{proj}_Y[i] : \mathbb{F}_2^n \to \mathbb{F}_2^{\delta_i+1}, \quad (x_1, \ldots, x_n) \mapsto (x_{j_1}, \ldots, x_i, \ldots, x_{j_{\delta_i}}). \tag{2}$$

The map projects from the full n-tuple x down to the states vertex i needs for updating its state. We set $x = (x_1, x_2, \ldots, x_n)$. For each $i \in \mathbb{N}_n$ there is a Y-local map $F_{i,Y} : \mathbb{F}_2^n \to \mathbb{F}_2^n$ given by

$$\begin{aligned} y_i &= f_{\delta_i+1} \circ \mathrm{proj}_Y[i], \\ F_{i,Y}(x) &= (x_1, \ldots, x_{i-1}, y_i(x), x_{i+1}, \ldots, x_n). \end{aligned} \tag{3}$$

The function $F_{i,Y}$ updates the state of vertex i and leaves all other states fixed. We refer to the sequence $(F_{i,Y})_i$ as F_Y. Note that for each graph $Y < K_n$ a sequence $(f_k)_{1 \le k \le n}$ induces a sequence F_Y, i.e. we have a map $\{Y < K_n\} \to \{F_Y\}$.

Definition 1 (Sequential Dynamical System) *Let $Y < K_n$ and let $(f_k)_k$ with $1 \le k \le d(Y) + 1$ be a sequence of symmetric functions as above. Let $\pi \in S_n$ where S_n denotes the symmetric group on n letters. Define the map*

$$[F_Y, \ ] : S_n \to \mathrm{Map}(\mathbb{F}_2^n, \mathbb{F}_2^n), \qquad [F_Y, \pi] = \prod_{i=1}^{n} F_{\pi(i),Y}, \tag{4}$$

*where product denotes composition. The sequential dynamical system (*SDS*) over Y induced by $(f_k)_k$ with respect to the ordering π is $[F_Y, \pi]$.*

We call an SDS homogeneous if it is induced by a sequence of local symmetric functions of the form $(f_k)_k = (B_k)_k$ where B is a Boolean function like, e.g. parity which returns the sum of its arguments modulo 2. By $[\mathbf{B}_Y, \pi]$ we mean the homogeneous SDS over Y induced by the Boolean function B.

Example 2 *Let $Y = \mathrm{Circ}_4$, the circle graph on 4 vertices. The graph is shown in figure 1. For each vertex we have a symmetric function on 3 arguments. To*

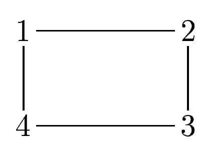

Fig. 1. The circle graph on 4 vertices, Circ_4.

be specific we pick the parity function on three arguments for each vertex. The function $\mathrm{par}_3 : \mathbb{F}_2^3 \to \mathbb{F}_2$ is defined by $\mathrm{par}_3(x_1, x_2, x_3) = \sum_i x_i \mod 2$. Thus

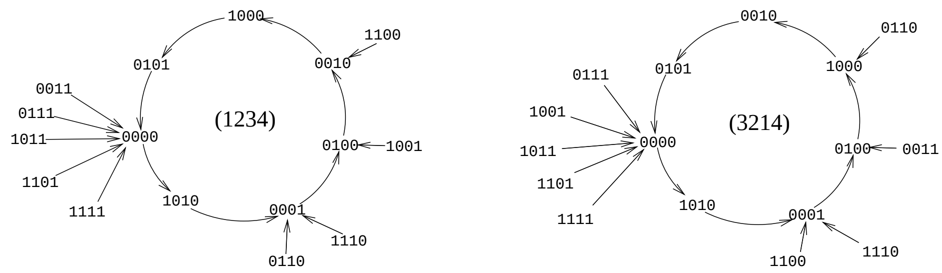

Fig. 2. For the update orders $\sigma = (1234)$ and $\pi = (3214)$ and with $F_i = \mathrm{Nor}_i$ the two SDS $[\mathbf{Nor}_{\mathrm{Circ}_4}, \sigma]$ (RHS) and $[\mathbf{Nor}_{\mathrm{Circ}_4}, \pi]$ (LHS) have non-identical phase spaces, but their digraphs are clearly isomorphic.

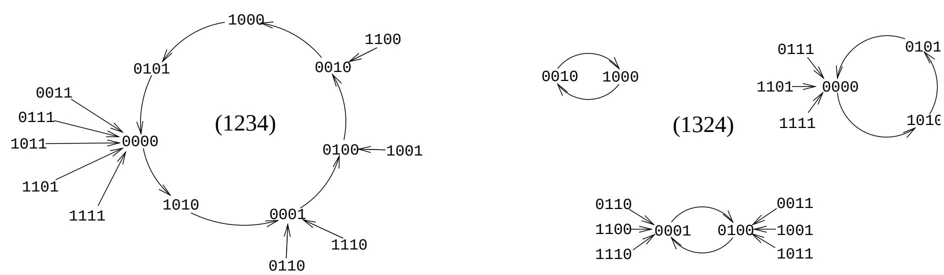

Fig. 3. The phase spaces of the two SDS $[\mathbf{Nor}_{Circ_4}, (1234)]$ and $[\mathbf{Nor}_{Circ_4}, (1324)]$ to the left and right, respectively. Clearly, the phase spaces are not identical, and their are also non-isomorphic.

for the update schedule $(1, 2, 3, 4)$ *with initial state* $(1, 1, 0, 0)$ *we get*

$$\begin{aligned} F_1(1,1,0,0) &= (0,1,0,0), \\ F_2 \circ F_1(1,1,0,0) &= (0,1,0,0), \\ F_3 \circ F_2 \circ F_1(1,1,0,0) &= (0,1,1,0), \\ F_4 \circ F_3 \circ F_2 \circ F_1(1,1,0,0) &= (0,1,1,1), \end{aligned}$$

and thus $[F_{\mathrm{Circ}_4}, (1,2,3,4)](1,1,0,0) = (0,1,1,1)$.

We introduce the equivalence relation $\sim_{Y,F}$ on $S_n \times S_n$ by $\pi \sim_{Y,F} \sigma$ iff $[F_Y, \pi] = [F_Y, \sigma]$, and we let $\mathbb{S}_{(f_k)_k}(Y) = \{[F_Y, \pi] \mid \pi \in S_n\}$. Note that we may have $\pi \sim_{Y,F} \sigma$ as a result of the structure of Y and/or the structure of the functions $F_{i,Y}$. Since phase space for an SDS is finite we may identify it with a finite unicyclic digraph.

Definition 3 *The digraph* $\Gamma[F_Y, \pi]$ *associated to the* SDS $[F_Y, \pi]$ *is the directed graph having vertex-set* $\mathbb{F}_2^n$ *and directed edges* $\{(x, [F_Y, \pi](x)) \mid x \in \mathbb{F}_2^n\}$.

The requirement that each f_k is to be symmetric is quite severe. In the next section we will discuss how we may relax this condition such that most results developed on SDS still remain valid.

2 A generalization of SDS

Restricting the local functions f_k to the class of symmetric functions is limiting for the possible SDS one can construct. Clearly, the main reason for insisting on having symmetric functions is to avoid the complication where the order of the states matters in the evaluation of f_k. In general, different orders of the arguments will produce different orbits of the dynamical system. One may argue that in many applications the local functions are symmetric. However, the fact that, e.g. the identity function is not in this class is a little discomforting. With the framework above it is impossible to have $F_{i,Y} = \mathrm{id}_{\mathbb{F}_2^n}$ for each vertex i unless the graph is the empty graph.

In [7, 8] we established a best possible upper bound for the number of functionally different SDS that can be obtained through changes of the update ordering while keeping the graph and the local functions fixed. Explicitly, there is a bijection

$$f_Y : S_n / \sim_Y \rightarrow \mathrm{Acyc}(Y)$$

where $\mathrm{Acyc}(Y)$ is the set of acyclic orientations of Y, and this bound is also sharp. This bound is also valid for non-symmetric functions. In fact, it is valid for arbitrary functions. Thus for giving the (best possible) upper bound for the number of functionally different SDS that can be obtained through rescheduling there is no reason for insisting on having symmetric functions.

In the following we will formulate SDS over quasi-symmetric functions and furthermore show that key results on SDS remain valid. For this purpose let E and F be vector spaces and let n be a positive integer. We consider E^n with the S_n-action $\pi(x_i) = (x_{\pi^{-1}(i)})$ and set

$$\mathrm{QSymm}(E^n, F) = \{f \in \mathrm{Map}(E^n, F) \mid \forall \sigma \in 1 \times S_{n-1} : f(\sigma x) = f(x)\}.$$

Similarly we write $\mathrm{Symm}(E^n, F)$ for the set of symmetric functions from E^n to F. If $E = F$ we simply write $\mathrm{Symm}(E^n)$ and $\mathrm{QSymm}(E^n)$. Accordingly, the elements of $\mathrm{QSymm}(\mathbb{F}_2^k)$ are the local functions that are symmetric in the neighbor states. Such functions are occasionally referred to as semi-totalistic rules in the literature on cellular automata. In order to formulate the SDS-framework for quasi-symmetric functions we set

$$\tilde{B}_{1,Y}(i) = (i, j_1, \dots, j_{\delta_i}) \; . \tag{5}$$

The state projection map is set accordingly

$$\mathrm{proj}_Y[i] : \mathbb{F}_2^n \to \mathbb{F}_2^{\delta_i+1}, \quad (x_1, \dots, x_n) \mapsto (x_i, x_{j_1}, \dots, x_{j_{\delta_i}}), \tag{6}$$

and the corresponding maps $F_{i,Y}$ are defined as before. We can now give:

Definition 4 (Sequential Dynamical System) *Let $Y < K_n$ and let $(f_k)_k$ with $1 \le k \le d(Y)+1$ be a sequence with $f_k \in \mathrm{QSymm}(\mathbb{F}_2^k)$. Let $\pi \in S_n$. The sequential dynamical system (*SDS*) over Y induced by $(f_k)_k$ with respect to the ordering π is $[F_Y, \pi]$.*

It is worth mentioning that there are significantly more quasi-symmetric than symmetric functions. We have e.g. $|\,\mathrm{QSymm}(\mathbb{F}_2^k)| = 2^{2k}$ and $|\,\mathrm{Symm}(\mathbb{F}_2^k)| = 2^{k+1}$. In particular, the functions $\mathrm{id}_k : \mathbb{F}_2^n \to \mathbb{F}_2$, $\mathrm{id}_k(x_i) = x_k$ are quasi-symmetric, whence the identity map can directly be considered as an SDS.

One of the main motivations for the use of symmetric functions was based on the study of equivalence of SDS. In general, two maps $F, G : \mathbb{F}_2^n \to \mathbb{F}_2^n$ are *dynamically equivalent* if there exists a bijection $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that

$$G = \varphi \circ F \circ \varphi^{-1}. \tag{7}$$

With the discrete topology on $\mathbb{F}_2^n$, of course, equation (7) says that F and G are topologically conjugated maps. In particular, (7) implies, that the two dynamical systems have a 1-1 correspondence between fixed points. For if x is fixed under F we obtain $G(\varphi(x)) = \varphi(x)$, i.e. $\varphi(x)$ is a fixed point for G. In order to demonstrate how the restriction on the local functions manifest themselves, we will give the proof of a conjugation result for SDS induced by quasi-symmetric functions.

Theorem 5 *Let $Y < K_n$ and define the S_n-action on $\mathbb{F}_2^n$ by $\rho(x) = (x_{\rho^{-1}(1)}, \dots, x_{\rho^{-1}(n)})$. For all $\pi \in S_n$ and all $\gamma \in \mathrm{Aut}(Y)$ we have*

$$[F_Y, \gamma\pi] = \gamma \circ [F_Y, \pi] \circ \gamma^{-1}. \tag{8}$$

PROOF. We can rewrite equation (8) as

$$\prod_{i=1}^{n} F_{\gamma\pi,Y} = \prod_{i=1}^{n} \gamma \circ F_{\pi(i),Y} \circ \gamma^{-1}.$$

We will show that the the ith factor on the left and the ith factor on the right gives the same result when applied to a state x. We have

$$F_{\gamma\pi(i),Y}(x) = (x_1, \ldots, \underbrace{f_{\gamma\pi(i)}(x_j \mid j \in B_1(\gamma\pi(i)))}_{\text{pos. } \gamma\pi(i)}, \ldots, x_n).$$

Similarly

$$\begin{aligned} \gamma \circ F_{\pi(i),Y} \circ \gamma^{-1}(x) &= \gamma \circ F_{\pi(i),Y}(x_{\gamma(1)}, \ldots, x_{\gamma(n)}) \\ &= \gamma(x_{\gamma(1)}, \ldots, \underbrace{f_{\pi(i)}(x_j \mid j \in \gamma B_1(\pi(i)))}_{\text{pos. } \pi(i)}, \ldots, x_{\gamma(n)}) \\ &= (x_1, \ldots, \underbrace{f_{\pi(i)}(x_j \mid j \in \gamma B_1(\pi(i)))}_{\text{pos. } \gamma\pi(i)}, \ldots, x_n). \end{aligned}$$

Equality follows from that fact that a graph automorphism γ of Y makes $\gamma B_1(\pi(i)) = B_1(\gamma\pi(i))$ and also preserves the center vertex, plus the fact that we have $f_{\pi(i)} = f_{\gamma\pi(i)}$ since by construction there is only one local function for a given degree.

We would like to point out that some earlier results on SDS will have to be modified. For instance, the classification of invertible SDS is no longer complete. We will return to this in section 4.

3 Fixed points of cellular automata

For SDS, and also for systems like sequential cellular automata, it is known that the fixed points are independent of the update order. To be precise we have:

Proposition 6 ([7]) *Let $Y < K_n$ and let $[F_Y, \pi]$ be an* SDS *over Y. We have*

$$\forall\, \sigma \in S_n : \qquad \mathrm{Fix}([F_Y, \sigma]) = \mathrm{Fix}([F_Y, \pi]). \tag{9}$$

A sequential cellular automaton (sCA) is constructed from a triple $(\mathrm{Circ}_n, \phi, \pi)$ where Circ_n is the circle graph on n vertices, $\phi : \mathbb{F}_2^3 \to \mathbb{F}_2$ is some function and $\pi \in S_n$. Loosely speaking, an sCA is an SDS over Circ_n where arbitrary local functions are allowed. If the updating is done in parallel we obtain a parallel (or classical) cellular automaton (pCA). It is clear from Proposition 6 that the fixed points of an sCA and a pCA are the same as long as they use the same function ϕ. In [4] this fact was used to study the structure of fixed points of all sCA and pCA, and also to derive recursion relations for the number of fixed points. Similar results have been obtained in, e.g. [6] in the setting of cellular automata.

A function or rule $\phi : \mathbb{F}_2^3 \to \mathbb{F}_2$ is called a radius-1 rule. A radius-r rule is a map of the form $\phi : \mathbb{F}_2^{2r+1} \to \mathbb{F}_2$. Some convention has to be made about the order of arguments to ϕ. Here we compute the new value of the state of vertex i as $\phi(x_{i-2r}, \ldots, x_i, \ldots, x_{i+2r})$, where indices 0 and n etc. are identified. The fixed point result in [4] only applies to the case with radius-1 rules. Here we extend this result to the case of cellular automata with radius-r rules.
For this purpose we introduce the graph $C_{n,r}$ having the property that any radius-r rule on Circ_n can be identified with $C_{n,r}$-local function:

$$\begin{aligned} \mathrm{v}[C_{n,r}] &= \{1, 2, \ldots, n\} \\ \mathrm{e}[C_{n,r}] &= \bigcup_{i=1}^{n} \{\{i, i+l\} \mid 1 \leq l \leq r\}, \end{aligned} \tag{10}$$

where indices i and $i+n$ etc. are identified. As an illustration we have $C_{6,2}$ in figure 4 below:

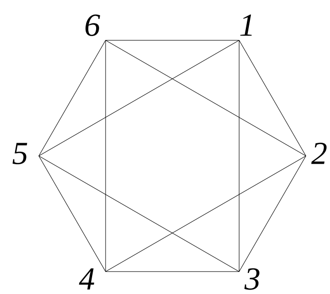

Fig. 4. The graph $C_{6,2}$.

Definition 7 *Let r be a positive integer and let $x, x' \in \mathbb{F}_2^{2r+1}$. We say that x is compatible with x' if $x_{i+1} = x'_i$, $1 \leq i \leq r-1$, and we write this as*

$x \rhd x'$. A sequence $C = (x^i \in \mathbb{F}_2^{2r+1})_{i=1}^n$ is a compatible covering of $C_{n,r}$ if $x^1 \rhd x^2 \rhd \cdots \rhd x^n \rhd x^1$.

Let $\phi_r : \mathbb{F}_2^{2r+1} \to \mathbb{F}_2$. A compatible covering $C = (x^i)_{i=1}^n$ of $C_{n,r}$ is a compatible fixed point covering with respect to ϕ_r if $\phi_r(x^i) = x^i_{r+1}$ for $1 \leq i \leq n$. The set of all compatible fixed point coverings of $C_{n,r}$ with respect to ϕ_r is denoted $\mathfrak{C}_\phi(n,r)$.

Let $\Phi_{C_{n,r}} : \mathrm{Map}(\mathbb{F}_2^{2r+1}, \mathbb{F}_2) \to$ **DiGraph** be the map assigning to ϕ_r the digraph Γ_{ϕ_r} given by

$$\begin{aligned} \mathrm{v}[\Gamma_{\phi_r}] &= \{x \in \mathbb{F}_2^{2r+1} \mid \phi_r(x) = x_{r+1}\}, \\ \mathrm{e}[\Gamma_\phi] &= \bigcup_{x \in \mathrm{v}[\Gamma_\phi]} \{(x, x') \mid x' \in \mathrm{v}[\Gamma_\phi],\ x \rhd x'\}. \end{aligned} \tag{11}$$

Note that a cycle of length n in Γ_{ϕ_r} corresponds to a compatible fixed point covering of $C_{n,r}$. Clearly, Γ_{ϕ_r} has at most 2^{2r+1} vertices. Each $C \in \mathfrak{C}_{\phi_r}(n,r)$ corresponds uniquely to a fixed point of an induced SCA (or PCA). To be precise define

$$\varphi : \mathfrak{C}_\phi(n,r) \to \mathrm{Fix}\,(\phi, C_{n,r}), \quad \varphi(x^1, x^2, \ldots, x^n) = (x^1_{r+1}, x^2_{r+1}, \ldots, x^n_{r+1}). \tag{12}$$

The map φ is one-to-one by construction.

Theorem 8 *Let $\phi \in \mathrm{Map}(\mathbb{F}_2^{2r+1}, \mathbb{F}_2)$. The number of fixed points L_n of an induced* SCA *over $C_{n,r}$ equals $|\mathfrak{C}_\phi(n,r)|$. Let A be the adjacency matrix of Γ_{ϕ_r}. We have*

$$L_n = \mathrm{Tr}\, A^n. \tag{13}$$

Let $\chi_A(l) = \sum_{i=0}^k a_i l^{k-i}$ be the characteristic polynomial of A. The number of fixed points L_n satisfies the recursion relation

$$\sum_{i=0}^k a_i L_{n-i} = 0. \tag{14}$$

PROOF. The first statement follows from the fact that φ is one-to-one and that $[A^n]_{ii}$ is the number of cycles of length n starting at vertex i. Now, (13) can be rewritten as

$$L_n = \mathrm{Tr}\, A^n = \sum_{i=0}^k e_i A^n e_i^T,$$

where e_i denotes the ith unit vector. The LHS of (14) now becomes

$$\begin{aligned}\sum_{i=0}^{k} a_i L_{n-i} &= \sum_{i=0}^{k} a_i (\sum_{j=0}^{k} e_j A^{n-i} e_j^T) \\ &= \sum_{j=0}^{k} (\sum_{i=0}^{k} e_j a_i A^{n-i} e_j^T) \\ &= \sum_{j=0}^{k} e_j (a_0 A^n + a_1 A^{n-1} + \cdots + a_k A^{n-k}) e_j^T \\ &= \sum_{j=0}^{k} e_j \chi_A(A) A^{n-k} e_j^T \\ &= 0,\end{aligned}$$

where the last equality follows from the Hamilton-Cayley theorem.

Example 9 (Majority) *For an* SDS *over* $C_{n,2}$ *induced by* maj_5 *we get the following vertices for* Γ_{maj_5}*:*

Hamming class	*Vertices*
0	(00000)
1	(00001), (00010), (01000), (10000)
2	(11000), (10010), (10001), (01010), (01001), (00011)
3	(11100), (10101), (00111), (01110), (01101), (10110)
4	(11110), (11101), (10111), (01111)
5	(11111)

The graph Γ_{maj_5} *is shown in figure 5 . By ignoring states such as* (11101) *that are "absorbing" in the sense that there can be no closed path containing them, we obtain for the reduced graph* $\chi(r) = r^{14} - 2r^{13} + 2r^{11} - r^{10} - r^8 + r^6$ *as the characteristic polynomical of the corresponding adjacency matrix. Thus the number of fixed points of an* SDS *induced by* $\mathrm{maj}_5 : \mathbb{F}_2^5 \to \mathbb{F}_2$, L_n, *satisfies*

$$L_n = 2L_{n-1} - 2L_{n-3} + L_{n-4} + L_{n-6} - L_{n-8},$$

and we have $L_5 = 2$, $L_6 = 10$, $L_7 = 16$, $L_8 = 28$, $L_9 = 38$, $L_{10} = 54$, $L_{11} = 68$, $L_{12} = 94$, $L_{13} = 132$.

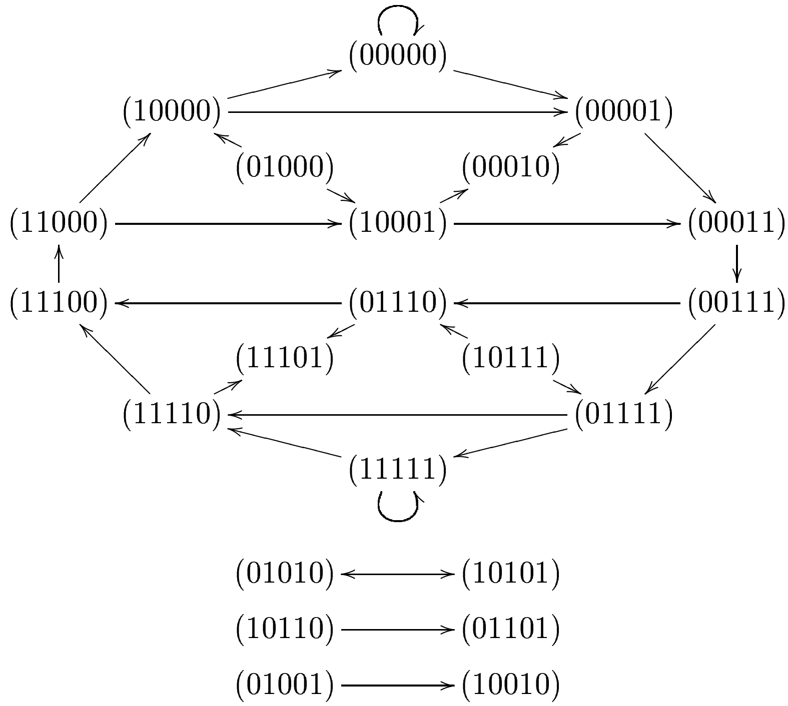

Fig. 5. The graph Γ_{maj_5}.

Example 10 (Parity) *In analogy to the above we obtain the graph* Γ_{par_5} *presented in figure 6. Accordingly, an* SDS *induced by* par_5 *over* $C_{n,2}$ *has* 16 *fixed*

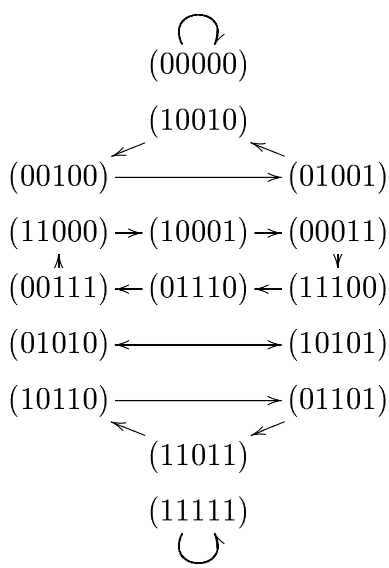

Fig. 6. The graph Γ_{par_5}.

points when $n \equiv 0 \pmod 6$, 8 *fixed points if* $n \equiv 0 \pmod 3$ *and* $n \not\equiv 0 \pmod 2$, 4 *fixed points if* $n \equiv 0 \pmod 2$ *and* $n \not\equiv 0 \pmod 3$ *and* 2 *fixed points otherwise.*

4 Invertible SDS

In [4] we characterized all sequences of symmetric local functions that induce invertible SDS. Here we will extend this result to also incorporate quasi-symmetric local functions. To begin we recall some results from [4, 7].

Proposition 11 *Let $Y < K_n$, let $(f_k)_k$ be a sequence $f_k \in \mathrm{Symm}(\mathbb{F}_2^k, \mathbb{F}_2)$ and let $\mathrm{id}, \mathrm{inv} : \mathbb{F}_2 \to \mathbb{F}_2$ be the maps defined by $\mathrm{id}(x) = x$ and $\mathrm{inv}(x) = \overline{x}$. An SDS $[F_Y, \pi]$ induced by $(f_k)_k$ is bijective if and only if for each $1 \leq i \leq n$ and fixed coordinates $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ the map*

$$g_i = f_{\delta_{i+1}, Y} \circ \mathrm{proj}_Y[i](x_1, \ldots, x_{i-1}, \ , x_{i+1}, \ldots, x_n) : \mathbb{F}_2 \to \mathbb{F}_2 \tag{15}$$

has the property $g_i \in \{\mathrm{id}, \mathrm{inv}\}$. Furthermore let $\pi = (i_1, \ldots, i_{n-1}, i_n) \in S_n$, $\pi^* = (i_n, i_{n-1}, \ldots, i_1)$ and $[F_Y, \pi]$ be a bijective SDS. Then we have

$$[F_Y, \pi]^{-1} = [F_Y, \pi^*].$$

Remark 12 *The above result is not limited to symmetric functions – it applies to arbitrary local functions. Also note that the inverse of an invertible SDS is an SDS. This holds for arbitrary local functions as well, but is generally false for, e.g. cellular automata.*

Define the function par_k by

$$\mathrm{par}_k : \mathbb{F}_2^k \to \mathbb{F}_2, \quad \mathrm{par}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i, \tag{16}$$

and the function $\overline{\mathrm{par}}_k : \mathbb{F}_2^k \to \mathbb{F}_2$ by $\overline{\mathrm{par}}_k(x_1, \ldots, x_k) = 1 - \mathrm{par}_k(x_1, \ldots, x_k)$.

Theorem 13 *Let Y be a graph an let $[F_Y, \pi]$ be an SDS over Y induced by a sequence of symmetric functions. Then the following assertions are equivalent:*
(i) $\forall\ \pi \in S_n$; $[F_Y, \pi]$ is invertible
(ii) $F_Y = (F_{i,Y})_i$ where $F_{i,Y} = \mathrm{Par}_{i,Y}$ or $F_{i,Y} = \overline{\mathrm{Par}}_{i,Y}$.

The proof uses the condition (15) in Proposition 11 and it is shown by induction that the value of a local symmetric function f_k on the state (0) determines f_k completely. As a consequence we see that there are (up to scheduling) only two homogeneous invertible SDS induced by symmetric local functions.

Let $f_k \in \mathrm{QSymm}(\mathbb{F}_2^k)$. In the following we will describe the structure imposed on f_k by requiring that each $F_{i,Y}$ is invertible. Let $x = (x_1, x_2, \ldots, x_k)$. Since $f_k \in \mathrm{QSymm}(\mathbb{F}_2^k)$ it only depends on the values $x_2, \ldots, x_k$ through their sum $s = \sum_{i=2}^k x_k$ (computed in $\mathbb{Z}$). Let $\bar{x} \in \mathbb{F}_2^{k-1}$ denote the $(k-1)$-tuple $(x_2, \ldots, x_k)$. Set $f(0, \bar{x}) = a_{0,s}$ and $f(1, \bar{x}) = a_{1,s}$. The condition (15) implies that $a_{0,s} = \overline{a_{1,s}}$. Thus we have:

Proposition 14 *Let $[F_Y, \pi]$ be an* SDS *over $Y < K_n$ induced by $(f_k)_k$, $f_k \in$ $\mathrm{QSymm}(\mathbb{F}_2^k)$. Then we have*

$$[F_Y, \pi] \textit{ is invertible } \Leftrightarrow \forall k \in \mathbb{N}_{d+1}\ \forall s \in \mathbb{Z}_k : [a_{0,s} = \overline{a_{1,s}}].$$

5 Sharpness of the Combinatorial Bound

In the introduction we already discussed the classification problem of simulations and that it is of interest to distinguish the phase space of two SDS up to isomorphism. In the classical theory of dynamical systems this problem is addressed in the Hartmann–Grobmann theorem which roughly states that an ODE and its linearization have the same behavior around a critical point provided that no eigenvalues have real part zero – their phase spaces are homeomorphic in an open set containing the critical point.

We say that two SDS $[F_Y, \pi]$ and $[G_Y, \sigma]$ are dynamically equivalent if there is a bijection $\phi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that

$$[F_Y, \pi] \circ \phi = \phi \circ [G_Y, \sigma]. \tag{17}$$

In fact, Proposition 5 relates the dynamics of two SDS that only differ by their update schedules. This naturally leads to the question: How many non-equivalent SDS can we obtain by varying the update order while keeping the local functions and the graph fixed?

In [9] the following framework has been introduced: A graph Y and an automorphism γ of Y induce a new (reduced) graph $\langle\gamma\rangle \setminus Y$ given by

$$\mathrm{v}[\langle\gamma\rangle \setminus Y] = \{\langle\gamma\rangle(i) \mid i \in \mathrm{v}[Y]\} \text{ and } \mathrm{e}[\langle\gamma\rangle \setminus Y] = \{\langle\gamma\rangle(y) \mid y \in \mathrm{e}[Y]\}.$$

Moreover, the $\mathrm{Aut}(Y)$-action on the vertex-set naturally induces an $\mathrm{Aut}(Y)$-

action on acyclic orientations given by

$$\{\gamma\,\mathfrak{O}\}(\{i,k\}) = \mathfrak{O}(\{\gamma^{-1}(i),\gamma^{-1}(k)\}).$$

We can now give the following upper bound $\Delta(Y)$ for the number of non-equivalent SDS [7, 9]:

$$\Delta(Y) = \frac{1}{|\operatorname{Aut}(Y)|}\sum_{\gamma\in\operatorname{Aut}(Y)} a(\langle\gamma\rangle\setminus Y) = \frac{1}{|\operatorname{Aut}(Y)|}\sum_{\gamma\in\operatorname{Aut}(Y)} |\operatorname{Fix}(\gamma)| \tag{18}$$

Here $a(Y)$ denotes the number of acyclic orientations of Y, and $\operatorname{Fix}(\gamma)$ is the set of acyclic orientations of Y that are fixed under γ. The quantity $\Delta(Y)$ is the number of orbits in $\operatorname{Acyc}(Y)$ under $\operatorname{Aut}(Y)$.

It is still an open question for which classes of graphs Y and which sequences of local functions (f_k) this bound is sharp. In [7] it is shown that the upper bound is always realized for Nor-systems over Star_n-graphs, where Star_n is defined by

$$\begin{aligned}\mathrm{v}[\mathrm{Star}_n] &= \{0,1,2,\ldots,n\},\\ \mathrm{e}[\mathrm{Star}_n] &= \{\{0,i\} \mid 1\le i\le n\}.\end{aligned}$$

In the following we will show that the bound is sharp for $\mathrm{Star}_{l,m}$-graphs. Let $\mathrm{Star}_{l,m}$ be the graph derived from K_l by joining to each of its vertices m other vertices. That is,

$$\begin{aligned}\mathrm{v}[\mathrm{Star}_{l,m}] &= \mathrm{v}[K_l] \cup \bigcup_{i=1}^{l}\{i_r \mid 1\le r\le m\} \qquad (19)\\ \mathrm{e}[\mathrm{Star}_{l,m}] &= \mathrm{e}[K_l] \cup \bigcup_{i=1}^{l}\{\{i,i_r\} \mid 1\le r\le m\}.\end{aligned}$$

The graph $\mathrm{Star}_{3,2}$ is shown in figure 7.

Lemma 15 *Let $m,l\ge 2$. We have*

$$\operatorname{Aut}(\mathrm{Star}_{l,m}) \cong S_m^l \rtimes S_l. \tag{20}$$

PROOF. An element γ of $\operatorname{Aut}(\mathrm{Star}_{l,m})$ maps K_l vertices in $\mathrm{Star}_{l,m}$ into K_l vertices since it is degree preserving. Since γ also preserves adjacency the degree-1 vertices i_l attached to vertex i can only be permuted among

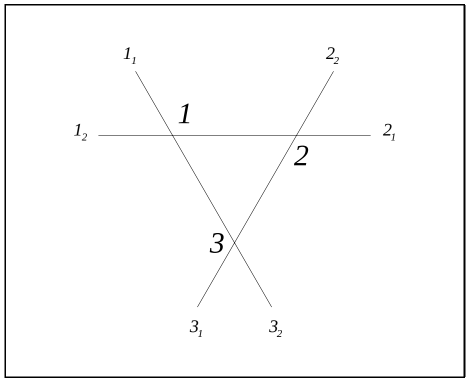

Fig. 7. The graph $\text{Star}_{3,2}$.

themselves and moved such that they are adjacent to $\gamma(i)$. Thus we see that $\text{Aut}(\text{Star}_{l,m}) = KH = HK$ where $H, K < S_{l(1+m)}$ are the groups

$$K = \left\{ \begin{pmatrix} 1 & 1_1 & \dots & 1_m & \dots & l & l_1 & \dots & l_m \\ 1 & \sigma_{1,1} & \dots & \sigma_{1,m} & \dots & l & \sigma_{l,1} & \dots & \sigma_{l,m} \end{pmatrix} \mid \sigma_i \in S(i_1, \dots, i_m) \right\} \tag{21}$$

and

$$H = \{ \sigma \in S_{l(m+1)} \mid \sigma(i) = j \Rightarrow \forall k \in \mathbb{N}_m : \sigma(i_k) = j_k \}. \tag{22}$$

Let $k \in K$ and $g = h \cdot k_1 \in \text{Aut}(\text{Star}_{l,m})$. Then we have

$$\begin{aligned} g \cdot k \cdot g^{-1} &= h \cdot k_1 \cdot k \cdot k_1^{-1} \cdot h^{-1} \\ &= h \cdot k_2 \cdot h^{-1}, \end{aligned}$$

where $k_2 = k_1 \cdot k \cdot k_1^{-1}$. In view of $h \cdot k_2 \cdot h^{-1} \in K$ we derive $K \lhd G$, and consequently $G = K \rtimes H$ follows. Since $K \cong S_m^l$ and $H \cong S_l$ we are done.

Proposition 16

$$\Delta(\text{Star}_{l,m}) = (m+1)^l \ . \tag{23}$$

PROOF. We will establish equation (23) by computing the bound in (18) directly. First, we know from Lemma 15 that $|\,\text{Aut}(\text{Star}_{l,m})| = l! \times m!^l$. We write automorphisms as $\gamma = (\sigma_l, \pi_1, \dots, \pi_l)$, where σ_l is the permutation of the vertices of the K_l subgraph and π_i denotes the permutation of the vertices $i_1, \dots, i_m$. We observe that $\gamma \in \text{Aut}(\text{Star}_{l,m})$ does only contribute to the sum in (18) in the case $\sigma_l = \text{id}$. If this was not the case the reduced graph $\langle\gamma\rangle \setminus \text{Star}_{l,m}$ would contain at least one loop, and would thus not allow for any acyclic orientations. Now with $\sigma_l = \text{id}$ it is clear that $\langle\gamma\rangle \setminus \text{Star}_{l,m}$ will be the graph K_l with $\#(\pi_i)$ vertices attached to vertex i of K_l. Here $\#(\gamma)$ denotes the number of cycles in the the cycle decomposition of γ where cycles of length one are included. Thus the number of acyclic orientations of the reduced graph $\langle\gamma\rangle \setminus \mathrm{Star}_{l,m}$ in this case is $l! \times 2^{\#(\gamma)}$. We now get:

$$\begin{aligned}
\Delta(\mathrm{Star}_{l,m}) &= \frac{1}{|\,\mathrm{Aut}(\mathrm{Star}_{l,m})|} \sum_{\gamma \in \mathrm{Aut}(\mathrm{Star}_{l,m})} a(\langle\gamma\rangle \setminus \mathrm{Star}_{l,m}) \\
&= \frac{1}{|\,\mathrm{Aut}(\mathrm{Star}_{l,m})|} \sum_{\gamma} a(\langle\gamma = (\mathrm{id}, \pi_1, \ldots, \pi_l)\rangle \setminus \mathrm{Star}_{l,m}) \\
&= \frac{1}{l! \times m!^l} \cdot l! \cdot \Big(\sum_{\gamma \in \mathrm{Star}_m} 2^{\#(\gamma)} \Big)^l \\
&= \left(\frac{\sum_{\gamma \in \mathrm{Star}_m} 2^{\#(\gamma)}}{m!} \right)^l \\
&= (m+1)^l,
\end{aligned}$$

where the last equality follows by induction, and we are done.

Let $\Delta_F(Y)$ denote the number of non-equivalent SDS that can be obtained from the sequence of local functions $F = (F_i)_i$ over the graph Y.

Theorem 17 *We have*

$$\Delta(\mathrm{Star}_{2,k}) = \Delta_{\mathbf{Nor}}(\mathrm{Star}_{2,k}). \tag{24}$$

Furthermore let $\pi \in S_{2k+2}$. *The* SDS $[\mathbf{Nor}_{\mathrm{Star}_{2,k}}, \pi]$ *has exactly one periodic orbit of length* 3.

In other words, the combinatorial upper bound Δ is sharp for the graph family $\mathrm{Star}_{2,k}$.

PROOF. Obviously, it is sufficient to show that SDS induced on disjoint orbits of $U(\mathrm{Star}_{2k+2})/\sim_{\mathrm{Star}_{2,k}}$ under $\mathrm{Aut}(\mathrm{Star}_{2,k})$ are pairwise non-equivalent. We will show this in three steps. First we will derive representative update schedules for each of these orbits. Second we compute the in-degree of the zero state in each of these cases with Nor as local functions and third we will prove that all corresponding phase spaces are in fact non-isomorphic.

To begin, we immediately observe that $S_{2,1} \cong \mathrm{Line}_4$ and it is easily checked that the bound $\Delta(\mathrm{Line}_4) = 4$ is sharp in this case. In the following we can therefore assume that $k \geq 2$ holds.

Step 1. Let $\pi \in S_{2k+2}$ and set

$$A_1(\pi) = \{1_k \mid 1_k <_\pi 1\}, \qquad A_2(\pi) = \{1_k \mid 1_k >_\pi 1\},$$
$$B_1(\pi) = \{2_k \mid 2_k <_\pi 2\}, \qquad B_2(\pi) = \{2_k \mid 2_k >_\pi 2\}.$$

The possible update schedules for SDS over $\text{Star}_{2,k}$ can be categorized as follows:

Case I: $[A_1 = \emptyset \vee A_2 = \emptyset] \wedge [B_1 = \emptyset \vee B_2 = \emptyset]$
Case II: $[[A_1 = \emptyset \vee A_2 = \emptyset] \wedge [B_1, B_2 \neq \emptyset]] \vee [[A_1, A_2 \neq \emptyset] \wedge [B_1 = \emptyset \vee B_2 = \emptyset]]$
Case III: $A_1, A_2, B_1, B_2 \neq \emptyset$.
Note that these classes are invariant under $\text{Aut}(\text{Star}_{2,k})$. We will write x_1, x_2, y_1 and y_2 for the states associcated to the vertices contained in A_1, A_2, B_1 and B_2 respectively. We also set $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$.

Case I is in fact similar to the line graph on 4 vertices. There are 4 orbits in $U(\text{Star}_{2,k})$ under $\text{Aut}(\text{Star}_{2,k})$ and these orbits have representatives

$$\pi_{1a} = (A, 1, 2, B), \qquad \pi_{1b} = (A, 1, B, 2),$$
$$\pi_{1c} = (1, A, 2, B), \qquad \pi_{1d} = (1, A, B, 2),$$

where, e.g. $(A, 1, 2, B)$ denotes the permutation $(1_1, \ldots, 1_k, 1, 2, 2_1, \ldots,)$. Similarly, class II has 4 main representatives that each fall into $k-1$ subcategories. Thus there are $4(k-1)$ orbits and the main representatives are given by

$$\pi_{2a} = (A_1, 1, A_2, 2, B), \qquad \pi_{2b} = (A_1, 1, A_2, B, 2),$$
$$\pi_{2c} = (B, 2, A_1, 1, A_2), \qquad \pi_{2d} = (2, B, A_1, 1, A_2).$$

By, e.g. $(A_1, 1, A_2, 2, B)$ we denote the permutation

$$(1_{a_1}, \ldots, 1_{a_r}, 1, 1_{a_{r+1}}, \ldots, 1_{a_k}, 2, 2_1, \ldots, 2_k),$$

where $1_{a_1}, \ldots, 1_{a_r} \in A_1$, $1_{a_{r+1}}, \ldots, 1_{a_k} \in A_2$, $a_1 < \cdots < a_r$ and $a_{r+1} < \cdots < a_k$.

Finally, in class III there is one main representative given by

$$\pi_3 = (A_1, 1, A_2, B_1, 2, B_2)$$

with $(k-1)^2$ sub-cases, and where the notation is as explained above. We further note that, in accordance with proposition 16, we have

$$4 + 4(k-1) + (k-1)^2 = (k+1)^2.$$

Step 2. From [9] we know that the state (0) has maximal preimage size, or alternatively, the indegree of (0) in the associated digraph of an SDS is maximal. It it thus sufficient for us to show that in all the cases above the indegrees of (0) are pairwise different. We summarize all the information in the table on page 19 where we write $|A_1| = k_1$ and $|B_1| = k_2$. From the table on page 19 we see that all potentially conflicting cases (where the indegree of zero is not sufficient to conclude non-isomorphism) occur within case 3 (i) (two pairs k_1, k_2 and k_1', k_2' give the same indegree), in case $1a$ vs. 3 (ii) and in case $2a$ vs. $2d$ (iii).

Case	Indegree of zero	Preimages of zero
$\pi_{1a} = (A, 1, 2, B)$	$d_{1a} = 1 + 2^{k+1}$	$(\underline{x}10\underline{1})$, $(\underline{1}01\underline{1})$, $(\underline{x}11\underline{1})$
$\pi_{1b} = (A, 1, B, 2)$	$d_{1b} = 2^k(1 + 2^k)$	$(\underline{1}0\underline{y}1)$, $(\underline{x}1\underline{y}1)$
$\pi_{1c} = (1, A, 2, B)$	$d_{1c} = 4$	$(1\underline{1}0\underline{1})$, $(0\underline{1}0\underline{1})$, $(0\underline{1}1\underline{1})$, $(1\underline{1}1\underline{1})$
$\pi_{1d} = (1, A, B, 2)$	$d_{1d} = 2^{k+1}$	$(0\underline{1}\underline{y}1)$, $(1\underline{1}\underline{y}1)$
$\pi_{2a} = (A_1, 1, A_2, 2, B)$	$d_{2a} = 2 + 2^{k_1+1}$	$(\underline{1}0\underline{1}0\underline{1})$, $(\underline{x_1}1\underline{1}0\underline{1})$, $(\underline{1}0\underline{1}1\underline{1})$, $(\underline{x_1}1\underline{1}1\underline{1})$
$\pi_{2b} = (A_1, 1, A_2, B, 2)$	$d_{2b} = 2^k(1 + 2^{k_1})$	$(\underline{1}0\underline{1}\underline{y}1)$, $(\underline{x_1}1\underline{1}\underline{y}1)$
$\pi_{2c} = (B, 2, A_1, 1, A_2)$	$d_{2c} = 2^k(1 + 2^{k_1}) + 2^{k_1}$	$(\underline{y}1\underline{1}0\underline{1})$, $(\underline{1}0\underline{x_1}1\underline{1})$, $(\underline{y}1\underline{x_1}1\underline{1})$
$\pi_{2d} = (2, B, A_1, 1, A_2)$	$d_{2d} = 2 + 2^{k_1+1}$	$(0\underline{11}0\underline{1})$, $(1\underline{11}0\underline{1})$, $(0\underline{1}\underline{x_1}1\underline{1})$, $(1\underline{1}\underline{x_1}1\underline{1})$
$\pi_3 = (A_1, 1, A_2, B_1, 2, B_2)$	$d_3 = (1 + 2^{k_1})(1 + 2^{k_2})$	$(\underline{1}0\underline{11}0\underline{1})$, $(\underline{x_1}1\underline{11}0\underline{1})$, $(\underline{1}0\underline{1}\underline{y_1}1\underline{1})$, $(\underline{x_1}1\underline{1}\underline{y_1}1\underline{1})$

Step 3. The phase spaces for the cases (i), (ii) and (iii) are non-isomorphic.

Claim. For any $\pi \in S_n$, $[\mathbf{Nor}_{\mathrm{Star}\,2,k}, \pi]$ has exactly one periodic cycle of length 3.

Clearly, it is enough to investigate a representative schedule from each of the classes above. We show this for case $1a$. The proof of the remaining cases is essentially the same. Here we have $\pi = (A, 1, 2, B)$. Assume the state $\chi = (X, x, y, Y)$ is a periodic point of period 3. If $x = 1$ then we must have $X = \underline{0}$ and $y = 0$. If $Y \neq \underline{0}, \underline{1}$ χ will have an even period whence $Y = \underline{0}$ or $Y = \underline{1}$ holds. This gives

$$\begin{aligned}
&(\underline{1}, 0, 0, \underline{0}) \mapsto (\underline{0}, 1, 1, \underline{0}) \mapsto (\underline{0}, 0, 0, \underline{1}) \mapsto (\underline{1}, 0, 0, \underline{0}) \\
&(\underline{1}, 0, 0, \underline{1}) \mapsto (\underline{0}, 1, 0, \underline{0}) \mapsto (\underline{0}, 0, 1, \underline{0}) \mapsto (\underline{0}, 1, 0, \underline{1}),
\end{aligned}$$

and accordingly there exists at least one periodic cycle of length 3. Similarly, we see that if $x = 0$ then $X = \underline{0}$ or $X = \underline{1}$ as χ will have an even period otherwise. Analogously we derive that $y = 1$ implies $Y = \underline{0}$ and if $y = 0$ we necessarily have $Y = \underline{1}$. This leaves us with the candidates $(\underline{0}, 1, 0, \underline{0})$, $(\underline{0}, 1, 0, \underline{1})$, $(\underline{0}, 1, 1, \underline{0})$, $(\underline{0}, 0, 0, \underline{0})$, $(\underline{0}, 0, 0, \underline{1})$ and $(\underline{0}, 0, 1, \underline{0})$, out of which only $(\underline{0}, 0, 0, \underline{0})$ is not covered above, but $(\underline{0}, 0, 0, \underline{0})$ maps to $(\underline{1}, 0, 0, \underline{1})$ which has period at least 4, proving the claim.

As a result of the above arguments we can determine the 3-cycle in case $1a$, $2a$, $2d$ and 3 as follows:

Case 1a:

$$\begin{tikzcd}
(\underline{x}10\underline{0}) \arrow[r] & (\underline{0}01\underline{0}) \arrow[r] & (\underline{1}00\underline{1}) \arrow[dl] \\
& (\underline{0}10\underline{0}) \arrow[u] &
\end{tikzcd}, \qquad x \neq \underline{0} \tag{25}$$

Case 2a:

$$\begin{tikzcd}
(\underline{x_1}1\underline{0}0\underline{0}) \arrow[r] & (\underline{0}0\underline{1}1\underline{0}) \arrow[r] & (\underline{1}0\underline{0}0\underline{1}) \arrow[dl] \\
& (\underline{0}1\underline{0}0\underline{0}) \arrow[u] &
\end{tikzcd}, \qquad x_1 \neq \underline{0} \tag{26}$$

Case 2d:

$$\begin{tikzcd}
(1\underline{1}\underline{x_1}1\underline{0}) \arrow[r] & (0\underline{00}0\underline{1}) \arrow[r] & (1\underline{01}0\underline{0}) \arrow[dl] \\
(0\underline{1}\underline{x_1}1\underline{0}) \arrow[ur] & (0\underline{10}1\underline{0}) \arrow[u] &
\end{tikzcd}, \qquad x_1 \neq \underline{0} \text{ in } (0\underline{1}\underline{x_1}1\underline{0}) \tag{27}$$

Case 3:

$$\begin{tikzcd}
(\underline{0}0\underline{1}\underline{y_1}1\underline{0}) \arrow[r] & (\underline{1}0\underline{00}0\underline{1}) \arrow[r] & (\underline{0}1\underline{01}0\underline{0}) \arrow[dl] \\
(\underline{x_1}1\underline{01}0\underline{0}) \arrow[r] & (\underline{0}0\underline{10}1\underline{0}) \arrow[u] &
\end{tikzcd}, \qquad x_1, y_1 \neq \underline{0} \tag{28}$$

ad (i): Diagram (28) immediately implies that there can be no two pairs of values (k_1, k_2) and (k_1', k_2') yielding the same indegree of $\underline{0}$ and isomorphic 3-cycles, whence in (i) we have non-isomorphic phase spaces.

ad (ii): Diagram (25) and (28) show that for (1a)-phase spaces only one vertex of the 3-cycle has indegree larger than one, while (2a) and (2d)-phase spaces always exhibit two vertices on the 3-cycle with indegree exceeding one, whence (ii) follows.

ad (iii): If (2a) and (2d) phase spaces exhibit the same indegree of zero, then the value of k_1 must be the same. Then it follows that the indegree of the 3-cycles in (26) and (27) are not equal and we are done.

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